Pricing Convertible Bonds:
A Laplace-Carlson Transform Approach

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1 Introduction

Companies have financed by issuance of new stocks or bonds issue. However, recently financing by convertible bonds issue has increased because the convertible bond is an attractive security for both the issuer and the investor. The convertible bond is a corporate debt security that gives the holder the right to forgo future coupon and principal payment and receive a prespecified number of common stock instead. In other words, the convertible bond is a hybrid security that combines characteristics with the straight bond and the American option. Therefore, the issuer can easily raise money because of lower yield rate. On the other hand, the investor can expect the stock appreciation in the faith value guarantee.

We should build in the characteristics with the straight bond and the American option to evaluate the price of the convertible bond. Especially, it is very important to focus on the possibility of voluntary conversion prior to maturity. We consider that the need to capture the characteristic gets more serious because the conversion to common stocks actually has increased every year. However, it is very difficult to calculate optimal conversion boundary analytically. In previous researches, the models of the convertible bond with the possibility of voluntary conversion prior to maturity are worked out numerically. It is very critical to take an interval time and a division of the underlying asset for pricing the convertible bond precisely. Due to calculating the price iteratively in previous researches, it takes much time and causes cumulative error.

We propose the closed form formula of the convertible bond with the possibility of voluntary conversion prior to maturity and by using the Laplace-Carlson transform. Consequently, the closed form formula can take less time to calculate and cause less error.

2 Existing Literature

2.1 Brennan and Schwartz [1]

Brennan and Schwartz [1] propose the convertible bond with the possibility of voluntary conversion prior to maturity and the call provision by the finite difference method.

They assume that asset value $V(t)$ follows a geometric Brownian motion

$$dV(t) = \mu V(t)dt + \sigma V(t)dW(t),$$

(1)

where $\mu$ is a instantaneous expected return rate of $V(t)$, $\sigma > 0$ is a instantaneous volatility of $V(t)$ and $W(t)$ is a standard Wiener process. Then, the convertible bond value $B(V(t), t)$ follows the partial differential equation

$$\frac{1}{2} \sigma^2 V^2(t) \frac{\partial^2 B}{\partial V^2} + rV(t) \frac{\partial B}{\partial V} - rB + \frac{\partial B}{\partial t} = 0,$$

(2)

where $r > 0$ is a instantaneous risk free rate of interest.

In addition, the bondary condition at maturity $T$ is

$$B(V(T), T) = \begin{cases} \gamma V(T) & \text{for } \gamma V(T) \geq F, \\ F & \text{for } Fl \leq V(T) \leq \frac{F}{\gamma}, \\ \frac{V(T)}{l} & \text{for } V(T) \leq Fl, \end{cases}$$

(3)

where $\gamma$ is a dilution factor, $F$ is a faith value and $l$ is the number of convertible bonds outstanding.

2.2 Kimura and Kikuchi [2]

Kimura and Kikuchi [2] propose the closed form formula of the installment option by using the Laplace-Carlson transform to change the partial
differential equation into the ordinary differential equation. They assume that stock price \( S(t) \) follows a geometric Brownian motion under risk neutral measure

\[
dS(t) = (r - \delta)S(t)dt + \sigma S(t)d\tilde{W}(t),
\]
where \( \delta \geq 0 \) is a constantaneous dividend rate of \( S(t) \) and \( \tilde{W}(t) \) is a standard Wiener process under risk neutral measure. Then, the installment call option value \( C(S(\tau), \tau) \) follows the partial differential equation

\[
(r - \delta)S(\tau) \frac{\partial C}{\partial \tau} + \frac{1}{2} \sigma^2 S^2(\tau) \frac{\partial^2 C}{\partial S^2} - rC + q
= \frac{\partial C}{\partial S} \quad \text{for } S(\tau) > S^*(\tau),
\]
where \( \tau = T - t \) is time to maturity, \( q > 0 \) is a constantaneous installment rate and \( S^*(\tau) \) is an optimal stopping boundary. Applying the Laplace-Carlson transform to equation (5), they obtain the ordinary differential equation

\[
(r - \delta)S \frac{d\tilde{C}}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 \tilde{C}}{dS^2} - r\tilde{C} - q
= \lambda \tilde{C} - \lambda (S - K)^+ \quad \text{for } S > \tilde{S}^*,
\]
where

\[
\tilde{C} = \int_0^\infty \lambda e^{-\lambda \tau} C(S(\tau), \tau)d\tau.
\]

From equation (6) and the boundary conditions, they derive closed form solutions for \( \tilde{C} \) and \( \tilde{S}^* \).

3 The Model

We focus on the possibility of voluntary conversion prior to maturity to solve analytically without considering the call provision or default prior to maturity. We assume that asset value \( V(t) \) follows a geometric Brownian motion under risk neutral measure

\[
dV(t) = (r - \delta)V(t)dt + \sigma V(t)d\tilde{W}(t).
\]
Then, the convertible bond value \( B(V(\tau), \tau) \) follows the partial differential equation

\[
(r - \delta)V(\tau) \frac{\partial B}{\partial \tau} + \frac{1}{2} \sigma^2 V^2(\tau) \frac{\partial^2 B}{\partial V^2} - rB
= \frac{\partial B}{\partial V} \quad \text{for } V(\tau) < V^*(\tau),
\]
where \( V^*(\tau) \) is a optimal conversion boundary.

We use the Laplace-Carlson transform to change the partial differential equation into the ordinary differential equation. Applying the Laplace-Carlson transform to equation (9), we obtain

\[
\int_0^\infty \lambda e^{-\lambda \tau} (r - \delta)V(\tau) \frac{\partial B}{\partial V}d\tau + \int_0^\infty \lambda e^{-\lambda \tau} \frac{1}{2} \sigma^2 V^2(\tau) \frac{\partial^2 B}{\partial V^2}d\tau - \int_0^\infty \lambda e^{-\lambda \tau} rB d\tau
= \int_0^\infty \lambda e^{-\lambda \tau} \frac{\partial B}{\partial \tau}d\tau \quad \text{for } V(\tau) < V^*(\tau),
\]
then solving the equation, we finally obtain the ordinary differential equation

\[
(r - \delta)V \frac{d\tilde{B}}{dV} + \frac{1}{2} \sigma^2 V^2 \frac{d^2 \tilde{B}}{dV^2} - r\tilde{B}
= \lambda \tilde{B} - \lambda \left[ \gamma V1_{\{\tilde{V} > \gamma\}} + F1_{\{FI < V < \gamma\}} \right]
\]

\[
+ \frac{V}{\gamma} 1_{\{V < FI\}} \quad \text{for } V < \tilde{V}^*,
\]
where

\[
\tilde{B} = \int_0^\infty \lambda e^{-\lambda \tau} B(V(\tau), \tau)d\tau,
\]
\[
\tilde{V}^* = \int_0^\infty \lambda e^{-\lambda \tau} V^*(\tau)d\tau.
\]
Solving the ordinary differential equation (11) with boundary conditions

\[
\lim_{V \uparrow V^*} \tilde{B} = \gamma \tilde{V}^*, \quad \lim_{V \downarrow 0} \frac{d\tilde{B}}{dV} = \gamma, \quad \lim_{V \downarrow 0} \tilde{B} = 0,
\]

and the continuity conditions of \( \tilde{B} \) and its first derivative at \( V = F/\gamma \) and \( V = FI \), we obtain closed form solutions for the Laplace-Carlson transform \( \tilde{V}^* \) and \( \tilde{B} \) as follows

\[
\tilde{V}^* = \left[ \frac{\delta \gamma (\theta_1 - 1)(\lambda + r)F^{\theta_2 - 1}}{\lambda r + \delta - r} \right]^{\frac{1}{\theta_2 - 1}},
\]

\[
\tilde{B} = \begin{cases}
\gamma V & \text{for } \tilde{V}^* \leq V, \\
A_1 V^{\theta_1} + A_2 V^{\theta_2} + \frac{\lambda \gamma}{\lambda + \sigma} V & \text{for } F/\gamma \leq V < \tilde{V}^*, \\
A_3 V^{\theta_1} + A_4 V^{\theta_2} + \frac{\lambda}{\lambda + \sigma} F & \text{for } FI < V < F/\gamma, \\
A_5 V^{\theta_1} + \frac{1}{\theta_2 (\lambda + \sigma)} V & \text{for } V \leq FI,
\end{cases}
\]

for \( \theta_1, \theta_2 \) are constants.
where

\[
A_1 = -\frac{\theta_2 \lambda (\lambda + r + (\delta - r) \theta_1)(\gamma \theta_2 - l^{-\theta_2})}{\theta_1 (\theta_1 - \theta_2)(\lambda + \delta)(\lambda + r)F_{\theta_2 - 1}^{-1}} \times \nabla^{*} \theta_2 - \theta_1 + \frac{\delta \gamma}{\theta_1 (\lambda + \delta)} \nabla^{*} 1 - \theta_1, \quad (17)
\]

\[
A_2 = \frac{\lambda (\lambda + r + (\delta - r) \theta_1)(\gamma \theta_2 - l^{-\theta_2})}{(\theta_1 - \theta_2)(\lambda + \delta)(\lambda + r)F_{\theta_2 - 1}^{-1}}, \quad (18)
\]

\[
A_3 = \frac{\lambda (\lambda + r + (\delta - r) \theta_1)(\gamma \theta_2 - l^{-\theta_2})}{(\theta_1 - \theta_2)(\lambda + \delta)(\lambda + r)} \times \frac{\delta \gamma}{\theta_1 (\lambda + \delta)} \nabla^{*} 1 - \theta_1, \quad (19)
\]

\[
A_4 = \frac{\theta_1 (r - \delta) - (\lambda + r) \lambda (F(\theta_1 - \theta_2)(\lambda + \delta)(\lambda + r))}{(\theta_1 - \theta_2)(\lambda + \delta)(\lambda + r)} \times \frac{r - \delta + \frac{\sigma^2}{2}}{\sigma^2} + \frac{\sqrt{(r - \delta - \frac{\sigma^2}{2})^2 + 2\sigma^2(\lambda + r)}}{\sigma^2}, \quad (22)
\]

\[
\theta_1 = \frac{-r + \delta + \frac{\sigma^2}{2}}{\sigma^2} - \frac{\sqrt{(r - \delta - \frac{\sigma^2}{2})^2 + 2\sigma^2(\lambda + r)}}{\sigma^2}, \quad (23)
\]

\[
\theta_2 = \frac{-r + \delta + \frac{\sigma^2}{2}}{\sigma^2} - \frac{\sqrt{(r - \delta - \frac{\sigma^2}{2})^2 + 2\sigma^2(\lambda + r)}}{\sigma^2}. \quad (23)
\]

Using the relationship between the Laplace-Carlson transform and the Laplace transform

\[
\mathcal{L}[B(V(t), t)] = \lambda \mathcal{L}[B(V(t), t)], \quad (24)
\]

we calculate the optimal conversion boundary \( V^*(t) \) and the convertible bond value \( B(V(t), t) \) by Gaver-Stehfest method which is a numerical Laplace transform inversion because solutions for the Laplace-Carlson transform are so complicated that they can’t be analytically inverted.

4 Numerical Experiment

From Brennan and Schwartz [1], the basic parameter values are given as follows: \( r = 0.005, \quad \delta = 0.01, \quad \sigma = 0.30, \quad \gamma = 0.01, \quad F = 40, \quad l = 10 \) and \( T = 20 \).

4.1 Validation

We compare our model with Brennan and Schwartz [1]. We use Brennan and Schwartz which is modified as required to comply with our assumption due to discussion about effectiveness of our model.

Figure 1 illustrates the convertible bond value of our model and Brennan and Schwartz [1]. Table 1 shows the rate of deviation between our model and Brennan and Schwartz [1] and the time to calculate the convertible bond value of our model and Brennan and Schwartz [1]. From Figure 1 and Table 1, each rate of deviation between our model and the previous research is much lower. Therefore our model is precise because the previous model by the finite difference method can calculate a precise value if an interval time and a division of the underlying asset is taken smaller. In addition, each calculation time of our model is much shorter than each calculation time of the previous research. As a result, our model is more efficient than the previous research.

![Figure 1: The convertible bond value](image)

Table 1: Deviation rate and calculation time

<table>
<thead>
<tr>
<th>Asset value</th>
<th>500</th>
<th>1000</th>
<th>1500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value B-S</td>
<td>19.63</td>
<td>26.85</td>
<td>31.66</td>
</tr>
<tr>
<td>Our model</td>
<td>19.61</td>
<td>26.79</td>
<td>31.56</td>
</tr>
<tr>
<td>Rate of deviation [%]</td>
<td>0.01</td>
<td>0.19</td>
<td>0.34</td>
</tr>
<tr>
<td>Time [sec]</td>
<td>348</td>
<td>361</td>
<td>337</td>
</tr>
<tr>
<td>Our model</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

*1B-S represents Brennan and Schwartz [1].
4.2 Comparative statics

We discuss the comparative statics of the optimal conversion boundary and the convertible bond value.

Figure 2 illustrates the comparative statics of the convertible bond value with respect to asset value and volatility. As this figure indicates, an increase in the volatility could increase or decrease the convertible bond value. At lower asset value, the expected loss through default as for a straight bond could be higher than the expected gain from conversion. Therefore, an increase in the volatility decreases the convertible bond value. On the other hand, at higher asset value, the expected gain from conversion could be higher than the expected loss through default as for a straight bond. Therefore, an increase in the volatility increases the convertible bond value.

Figure 3 illustrates the comparative statics of the optimal conversion boundary with respect to volatility and time to maturity. This figure shows an increase in the volatility increases the optimal conversion boundary except the case of long time to maturity. This means that the investor expects asset value will be higher. Furthermore, this figure shows the optimal conversion boundary is a concave function of time to maturity. First, with long time to maturity, the optimal conversion boundary is lower due to an increase in the possibility of default. As time to maturity gets shorter, the optimal conversion boundary gets lower because the effect of the expected gain from conversion gets smaller. On the other hand, as this figure indicates, the optimal conversion boundary is much lower when the volatility is high and time to maturity is long. This means that the incentive which the holder converts it earlier increases because the possibility of default is much higher.

5 Conclusion

In this paper, we have proposed the closed form formula of the convertible bond with the possibility of voluntary conversion prior to maturity and by using the Laplace-Carlson transform. Moreover, comparing with the previous research by the finite difference method, we have shown our model gives the precise convertible bond value. Consequently, we can calculate the convertible bond value precisely in very short time.

6 Future Research

- Validation between our model and the market price
- Expansion into the model with call provision or default prior to maturity

References
